# V Vektoranalysis

## V.1 Vektor–Identitäten

$$(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a} \tag{V.1}$$

$$\nabla \wedge \nabla\Phi = 0 \tag{V.2}$$

$$\nabla \cdot (\nabla \wedge \mathbf{f}) = 0 \tag{V.3}$$

$$\nabla \cdot (\Phi\mathbf{f}) = \Phi\nabla \cdot \mathbf{f} + \mathbf{f} \cdot \nabla\Phi \tag{V.4}$$

$$\nabla \wedge (\Phi\mathbf{f}) = \Phi\nabla \wedge \mathbf{f} + (\nabla\Phi) \wedge \mathbf{f} \tag{V.5}$$

$$\nabla \wedge (\mathbf{f} \wedge \mathbf{g}) = (\mathbf{g} \cdot \nabla)\mathbf{f} - (\mathbf{f} \cdot \nabla)\mathbf{g} + \mathbf{f}(\nabla \cdot \mathbf{g}) - \mathbf{g}(\nabla \cdot \mathbf{f}) \tag{V.6}$$

$$\nabla \cdot (\mathbf{f} \wedge \mathbf{g}) = \mathbf{g} \cdot (\nabla \wedge \mathbf{f}) - \mathbf{f} \cdot (\nabla \wedge \mathbf{g}) \tag{V.7}$$

$$\nabla(\mathbf{f} \cdot \mathbf{g}) = \mathbf{f} \wedge (\nabla \wedge \mathbf{g}) + \mathbf{g} \wedge (\nabla \wedge \mathbf{f}) + (\mathbf{f} \cdot \nabla)\mathbf{g} + (\mathbf{g} \cdot \nabla)\mathbf{f} \tag{V.8}$$

$$(\mathbf{f} \cdot \nabla)\mathbf{f} = (\nabla \wedge \mathbf{f}) \wedge \mathbf{f} + \nabla(\frac{1}{2}\mathbf{f}^2) \tag{V.9}$$

$$\Delta\mathbf{f} = \nabla(\nabla \cdot \mathbf{f}) - \nabla \wedge (\nabla \wedge \mathbf{f}) \tag{V.10}$$

## V.2 Satz von Gauß

### Satz von Gauß

$$\int_A \mathbf{f} \cdot \mathbf{n}\, dA = \int_V \nabla \cdot \mathbf{f}\, dV \quad \text{bzw.} \quad \int_A f_j n_j\, dA = \int_V f_{j,j}\, dV \tag{V.11}$$

### Konsequenz 1

$$\int_A \Phi\mathbf{n}\, dA = \int_V \nabla\Phi\, dV \quad \text{bzw.} \quad \int_A \Phi n_j\, dA = \int_V \frac{\partial\Phi}{\partial x_j}\, dV \tag{V.12}$$

### Konsequenz 2

$$\int_A \sigma_{ij} n_j\, dA = \int_V \sigma_{ij,j}\, dV \tag{V.13}$$

### Konsequenz 3

$$\int_A \mathbf{f} \wedge \mathbf{n}\, dA = -\int_V \nabla \wedge \mathbf{f}\, dV \tag{V.14}$$

**Konsequenz 4**

$$\int_A (\nabla \Phi) \cdot \mathbf{n}\, dA = \int_V \Delta \Phi\, dV \tag{V.15}$$

**Konsequenz 5 (1. Greenscher Satz)**

$$\int_A \Phi \frac{\partial \Psi}{\partial n}\, dA = \int_V (\Phi \Delta \Psi + \nabla \Phi \cdot \nabla \Psi)\, dV \tag{V.16}$$

**Konsequenz 6 (2. Greenscher Satz)**

$$\int_A \left( \Phi \frac{\partial \Psi}{\partial n} - \Psi \frac{\partial \Phi}{\partial n} \right) dA = \int_V (\Phi \Delta \Psi - \Psi \Delta \Phi)\, dV \tag{V.17}$$

## V.3 Satz von Stokes

**Satz von Stokes**

$$\int_C \mathbf{f} \cdot d\mathbf{x} = \int_A (\nabla \wedge \mathbf{f}) \cdot \mathbf{n}\, dA \tag{V.18}$$

**Konsequenz 1**

$$\int_C u\, dx + v\, dy = \int_A \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx\, dy \tag{V.19}$$

**Konsequenz 2**

$$\int_C \Phi\, d\mathbf{x} = - \int_A (\nabla \Phi) \wedge \mathbf{n}\, dA \tag{V.20}$$

## V.4 Zylinderkoordinaten

**Umrechnung**

$$x = r \cos\varphi, \quad y = r \sin\varphi, \quad z = z \tag{V.21}$$

$$r = \sqrt{x^2 + y^2}, \quad \tan\varphi = \frac{y}{x}, \quad z = z \tag{V.22}$$

$$\mathbf{e}_r = \cos\varphi\, \mathbf{e}_x + \sin\varphi\, \mathbf{e}_y, \quad \mathbf{e}_\varphi = -\sin\varphi\, \mathbf{e}_x + \cos\varphi\, \mathbf{e}_y, \quad \mathbf{e}_z = \mathbf{e}_z \tag{V.23}$$

$$\mathbf{e}_x = \cos\varphi\, \mathbf{e}_r - \sin\varphi\, \mathbf{e}_\varphi, \quad \mathbf{e}_y = \sin\varphi\, \mathbf{e}_r + \cos\varphi\, \mathbf{e}_\varphi, \quad \mathbf{e}_z = \mathbf{e}_z \tag{V.24}$$

**Ableitung der Einheitsvektoren** Die ersten Ableitungen der Einheitsvektoren nach $r$, $\varphi$ und $z$ sind null bis auf

$$\frac{\partial \mathbf{e}_r}{\partial \varphi} = \mathbf{e}_\varphi, \quad \frac{\partial \mathbf{e}_\varphi}{\partial \varphi} = -\mathbf{e}_r \tag{V.25}$$

**Gradient**

$$\nabla \Phi = \frac{\partial \Phi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial \Phi}{\partial \varphi}\mathbf{e}_\varphi + \frac{\partial \Phi}{\partial z}\mathbf{e}_z \tag{V.26}$$

**Divergenz**

$$\nabla \cdot \mathbf{f} = \frac{1}{r}\frac{\partial}{\partial r}(r f_r) + \frac{1}{r}\frac{\partial f_\varphi}{\partial \varphi} + \frac{\partial f_z}{\partial z} \tag{V.27}$$

**Rotation**

$$\nabla \wedge \mathbf{f} = \frac{1}{r}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\varphi & \mathbf{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ f_r & r f_\varphi & f_z \end{vmatrix} \tag{V.28}$$

$$= \left(\frac{1}{r}\frac{\partial f_z}{\partial \varphi} - \frac{\partial f_\varphi}{\partial z}\right)\mathbf{e}_r + \left(\frac{\partial f_r}{\partial z} - \frac{\partial f_z}{\partial r}\right)\mathbf{e}_\varphi + \frac{1}{r}\left(\frac{\partial (f_\varphi r)}{\partial r} - \frac{\partial u_r}{\partial \varphi}\right)\mathbf{e}_z \tag{V.29}$$

**Laplace–Operator (skalar)**

$$\Delta \Phi = \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \varphi^2} + \frac{\partial^2 \Phi}{\partial z^2} \tag{V.30}$$

**Laplace–Operator (vektoriell)**

$$\Delta \mathbf{f} = \left(\Delta f_r - \frac{1}{r^2}\left[f_r + 2\frac{\partial f_\varphi}{\partial \varphi}\right]\right)\mathbf{e}_r + \left(\Delta f_\varphi - \frac{1}{r^2}\left[f_\varphi - 2\frac{\partial f_r}{\partial \varphi}\right]\right)\mathbf{e}_\varphi + (\Delta f_z)\mathbf{e}_z \tag{V.31}$$

**$\mathbf{u} \cdot \nabla$**

$$\mathbf{u} \cdot \nabla = u_r\frac{\partial}{\partial r} + \frac{u_\varphi}{r}\frac{\partial}{\partial \varphi} + u_z\frac{\partial}{\partial z} \tag{V.32}$$

**Divergenz eines Tensors**

$$\nabla \cdot \sigma = \left(\frac{1}{r}\frac{\partial(\sigma_{rr} r)}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\varphi r}}{\partial \varphi} + \frac{\partial \sigma_{zr}}{\partial z} - \frac{\sigma_{\varphi\varphi}}{r}\right)\mathbf{e}_r$$

$$+\left(\frac{1}{r}\frac{\partial(\sigma_{r\varphi}r)}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\varphi\varphi}}{\partial\varphi}+\frac{\partial\sigma_{z\varphi}}{\partial z}+\frac{\sigma_{r\varphi}}{r}\right)\mathbf{e}_\varphi \tag{V.33}$$
$$+\left(\frac{1}{r}\frac{\partial(\sigma_{rz}r)}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\varphi z}}{\partial\varphi}+\frac{\partial\sigma_{zz}}{\partial z}\right)\mathbf{e}_z$$

### V.4.1 Wärmeleitungsgleichung

#### Wärmefluss

$$\mathbf{q}=-k\nabla T=-k\left(\frac{\partial T}{\partial r}\mathbf{e}_r+\frac{1}{r}\frac{\partial T}{\partial\varphi}\mathbf{e}_\varphi+\frac{\partial T}{\partial z}\mathbf{e}_z\right) \tag{V.34}$$

#### Wärmeleitungsgleichung ($k=$ const)

$$\rho C\frac{\partial T}{\partial t}-k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 T}{\partial\varphi^2}+\frac{\partial^2 T}{\partial z^2}\right]=Q \tag{V.35}$$

### V.4.2 Navier–Stokes–Gleichung

#### Dehnraten–Tensor

$$e_{rr}=\frac{\partial u_r}{\partial r},\quad e_{\varphi\varphi}=\frac{1}{r}\frac{\partial u_\varphi}{\partial\varphi}+\frac{1}{r}u_r,\quad e_{zz}=\frac{\partial u_z}{\partial z} \tag{V.36}$$

$$2e_{r\varphi}=2e_{\varphi r} = r\frac{\partial(r^{-1}u_\varphi)}{\partial r}+\frac{1}{r}\frac{\partial u_r}{\partial\varphi} \tag{V.37}$$
$$2e_{rz}=2e_{zr} = \frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r} \tag{V.38}$$
$$2e_{\varphi z}=2e_{z\varphi} = \frac{1}{r}\frac{\partial u_z}{\partial\varphi}+\frac{\partial u_\varphi}{\partial z} \tag{V.39}$$

#### Kontinuitätsgleichung ($\rho\neq$ const)

$$\frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(\rho u_r r)+\frac{1}{r}\frac{\partial}{\partial\varphi}(\rho u_\varphi)+\frac{\partial}{\partial z}(\rho u_z)=0 \tag{V.40}$$

#### Kontinuitätsgleichung ($\rho=$ const)

$$\frac{1}{r}\frac{\partial(u_r r)}{\partial r}+\frac{1}{r}\frac{\partial u_\varphi}{\partial\varphi}+\frac{\partial u_z}{\partial z}=0 \tag{V.41}$$

**Navier–Stokes–Gleichung ($\rho, \eta = $ const)**

$$\rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + u_z\frac{\partial u_r}{\partial z} + \frac{1}{r}\left[u_\varphi\frac{\partial u_r}{\partial \varphi} - u_\varphi^2\right]\right) = -\frac{\partial p}{\partial r} + \eta\left(\Delta u_r - \frac{1}{r^2}\left[u_r + 2\frac{\partial u_\varphi}{\partial \varphi}\right]\right) \qquad (V.42)$$

$$\rho\left(\frac{\partial u_\varphi}{\partial t} + u_r\frac{\partial u_\varphi}{\partial r} + u_z\frac{\partial u_\varphi}{\partial z} + \frac{1}{r}\left[u_\varphi\frac{\partial u_\varphi}{\partial \varphi} + u_r u_\varphi\right]\right) = -\frac{1}{r}\frac{\partial p}{\partial \varphi} + \eta\left(\Delta u_\varphi - \frac{1}{r^2}\left[u_\varphi - 2\frac{\partial u_r}{\partial \varphi}\right]\right) \qquad (V.43)$$

$$\rho\left(\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + u_z\frac{\partial u_z}{\partial z} + \frac{1}{r}u_\varphi\frac{\partial u_z}{\partial \varphi}\right) = -\frac{\partial p}{\partial z} + g_z + \eta\Delta u_z \qquad (V.44)$$

## V.5 Kugelkoordinaten

**Umrechnung**

$$x = r\sin\vartheta\cos\varphi, \quad y = r\sin\vartheta\sin\varphi, \quad z = r\cos\vartheta \qquad (V.45)$$

$$r = \sqrt{x^2+y^2+z^2}, \quad \cos\vartheta = \frac{z}{\sqrt{x^2+y^2+z^2}}, \quad \tan\varphi = \frac{y}{x} \qquad (V.46)$$

$$\begin{aligned} \mathbf{e}_r &= \sin\vartheta\cos\varphi\mathbf{e}_x + \sin\vartheta\sin\varphi\mathbf{e}_y + \cos\vartheta\mathbf{e}_z \\ \mathbf{e}_\vartheta &= \cos\vartheta\cos\varphi\mathbf{e}_x + \cos\vartheta\sin\varphi\mathbf{e}_y - \sin\vartheta\mathbf{e}_z \\ \mathbf{e}_\varphi &= -\sin\varphi\mathbf{e}_x + \cos\varphi\mathbf{e}_y \end{aligned} \qquad (V.47)$$

$$\begin{aligned} \mathbf{e}_x &= \sin\vartheta\cos\varphi\mathbf{e}_r + \cos\vartheta\cos\varphi\mathbf{e}_\vartheta - \sin\varphi\mathbf{e}_\varphi \\ \mathbf{e}_y &= \sin\vartheta\sin\varphi\mathbf{e}_r + \cos\vartheta\sin\varphi\mathbf{e}_\vartheta - \cos\varphi\mathbf{e}_\varphi \\ \mathbf{e}_z &= \cos\vartheta\mathbf{e}_r - \sin\vartheta\mathbf{e}_\theta \end{aligned} \qquad (V.48)$$

**Ableitung der Einheitsvektoren** Die ersten Ableitungen der Einheitsvektoren nach $r$, $\vartheta$ und $\varphi$ sind null bis auf

$$\frac{\partial \mathbf{e}_r}{\partial \vartheta} = \mathbf{e}_\vartheta, \quad \frac{\partial \mathbf{e}_\vartheta}{\partial \vartheta} = -\mathbf{e}_r, \quad \frac{\partial \mathbf{e}_r}{\partial \varphi} = \sin\vartheta\mathbf{e}_\varphi, \quad \frac{\partial \mathbf{e}_\vartheta}{\partial \varphi} = \cos\vartheta\mathbf{e}_\varphi, \quad \frac{\partial \mathbf{e}_\varphi}{\partial \varphi} = \mathbf{e}_z \wedge \mathbf{e}_\varphi \qquad (V.49)$$

**Gradient**

$$\nabla\Phi = \frac{\partial \Phi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial \Phi}{\partial \vartheta}\mathbf{e}_\vartheta + \frac{1}{r\sin\vartheta}\frac{\partial \Phi}{\partial \varphi}\mathbf{e}_\varphi \qquad (V.50)$$

**Divergenz**

$$\nabla \cdot \mathbf{f} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 f_r) + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}(\sin\vartheta f_\vartheta) + \frac{1}{r\sin\vartheta}\frac{\partial f_\varphi}{\partial\varphi} \tag{V.51}$$

**Rotation**

$$\begin{aligned}
\nabla \wedge \mathbf{f} &= \frac{1}{r^2\sin\vartheta}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\vartheta & r\sin\vartheta\mathbf{e}_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\vartheta} & \frac{\partial}{\partial\varphi} \\ f_r & rf_\vartheta & (r\sin\vartheta)f_\varphi \end{vmatrix} \\
&= \frac{1}{r\sin\vartheta}\left(\frac{\partial}{\partial\vartheta}(\sin\vartheta f_\varphi) - \frac{\partial f_\vartheta}{\partial\varphi}\right)\mathbf{e}_r + \frac{1}{r}\left(\frac{1}{\sin\vartheta}\frac{\partial f_r}{\partial\varphi} - \frac{\partial}{\partial r}(rf_\varphi)\right)\mathbf{e}_\vartheta \\
&\quad + \frac{1}{r}\left(\frac{\partial}{\partial r}(rf_\vartheta) - \frac{\partial f_r}{\partial\vartheta}\right)\mathbf{e}_\varphi
\end{aligned} \tag{V.52}$$

**Laplace–Operator (skalar)**

$$\Delta\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial\Phi}{\partial\vartheta}\right) + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2\Phi}{\partial\varphi^2} \tag{V.53}$$

### V.5.1 Wärmeleitungsgleichung

**Wärmefluss**

$$\mathbf{q} = -k\nabla T = -k\left(\frac{\partial T}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial T}{\partial\vartheta}\mathbf{e}_\vartheta + \frac{1}{r\sin\vartheta}\frac{\partial T}{\partial\varphi}\mathbf{e}_\varphi\right) \tag{V.54}$$

**Wärmeleitungsgleichung ($k =$ const)**

$$\rho C\frac{\partial T}{\partial t} - k\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial T}{\partial\vartheta}\right) + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2 T}{\partial\varphi^2}\right] = Q \tag{V.55}$$